Tropical Algebra

SIG Math Fall 2025

Overview

- The main theme of today is that **when you have a hard problem in one domain, it's often easier to transform it to an easier problem in another domain.**
- In this case, we'll discuss some well-known problems from computer science and how they can be made easier by looking at them through the lens of tropical algebra.
- Another idea to keep in mind is that when you have a lot of similar structures, if you know something about one you often know something about many.

What do the following objects have in common?

- Integers
- Rational numbers
- Real numbers
- Complex numbers
- Polynomials
- Functions
- Matrices
- Booleans (true/false)
- Types

You can add and multiply them!

A brief history lesson

- Mathematics arguably started from trying to find solutions to polynomials
    - It's also what you spend a lot of time learning in high school!
- By the 19th century, **polynomials were very well-studied** and mathematicians knew a lot of ways to work with them.
- Around this time, calculus and linear algebra started to become developed, and drew heavily from theories about polynomials.
- In the 20th century, computers were just getting invented and computer scientists had to figure out the best way to solve problems such as:
    - Finding the shortest path in a graph (Google Maps)
    - What word or sentence is someone saying (voice recognition)
    - How closely related are two species' DNA (computational biology)
    - How to optimize neural networks (ChatGPT)

Semirings explained

- During the 19th century, mathematicians noted that **a lot of the objects that they studied had similar properties**, such as:
    - You can add things and multiply things to get something else
    - Addition and multiplication are commutative
    - Addition and multiplication are associative
    - Multiplication distributes over addition: $a(b + c) = ab + ac$
    - Multiplying by 0 equals 0, multiplying by 1 doesn't change anything
    - Adding with 0 doesn't change anything
    - There usually is a notion of subtraction, but not always
- Anything that satisfied these properties is called a **semiring.**
- What mathematicians realized is that if you only use properties of addition and multiplication, **if you proved something once it would apply to any semiring.**

So what is tropical algebra?

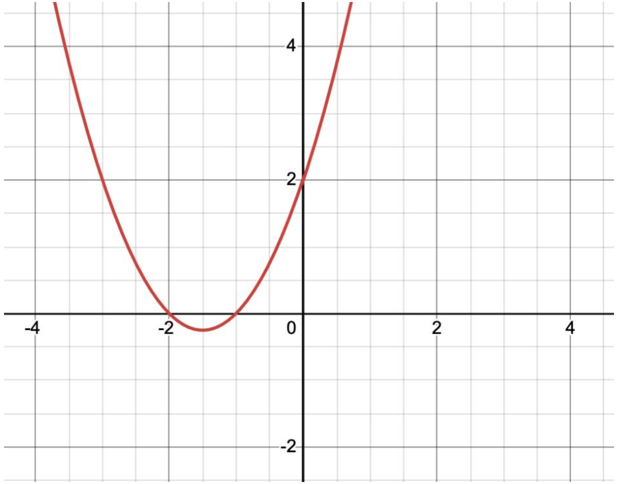
- It's just normal algebra where we change the meanings of + and *
- There are a few flavors, but they are all very similar:
  - Min-Plus:
    - Tropical addition (denoted by $\oplus$): take the minimum, e.g. $3 \oplus 7 = 3$
    - Tropical multiplication (denoted by $\otimes$): take the sum, e.g. $3 \otimes 7 = 10$
  - Max-Plus
    - Tropical addition (denoted by $\oplus$): take the maximum, e.g. $3 \oplus 7 = 7$
    - Tropical multiplication (denoted by $\otimes$): take the sum, e.g. $3 \otimes 7 = 10$
  - Max-Times
    - Tropical addition (denoted by $\oplus$): take the maximum, e.g. $3 \oplus 7 = 7$
    - Tropical multiplication (denoted by $\otimes$): take the product, e.g. $3 \otimes 7 = 21$
- We will be using max-plus by default, but I will always specify if we use something else.

Contrasting tropical and normal algebra

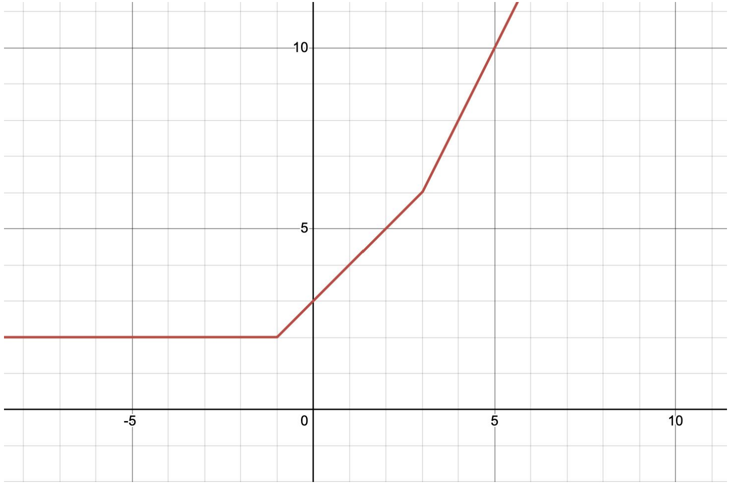
- Let's first look at the similarities:
  - Both systems have an additive and multiplicative identity
  - Both systems are commutative, associative, and distributive
  - If you multiply or add two elements, you get another element
- But there are a few problems. In normal algebra, adding 0 to a number gives you that number back. But if you do do $-5 \oplus 0$, you get 0, not -5. So **we need a different "zero value"**.
- The solution is to add a value of $-\infty$. Note that:
  - $-\infty \oplus n = n$ (since any number is greater than negative infinity)
  - $-\infty \otimes n = -\infty$ (in normal algebra any number times 0 is 0)
- So **$-\infty$ is our version of zero** in tropical algebra!
- Subtraction also isn't a thing in tropical algebra.

Polynomials in tropical algebra

- In normal algebra a polynomial would look like $x^2 + 3x + 2$
- In tropical algebra, it would look like: $(x \otimes x) \oplus (3 \otimes x) \oplus 2$



Normal graph of $x^2 + 3x + 2$



Tropical graph of $x^2 + 3x + 2$

Solving tropical polynomials

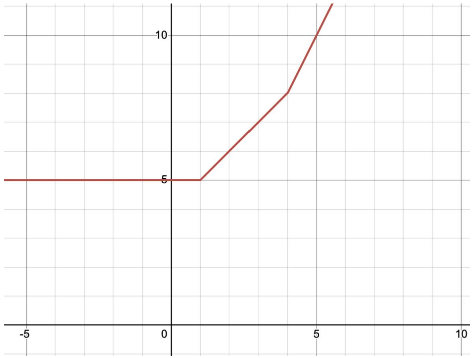
- Typically when we are given a polynomial, a natural thing to do is to try and solve it.
- Let’s be specific though: typically when solving the polynomial, we mean finding the values of x where the polynomial equals zero. However, remember that in tropical algebra, -∞ takes the role of zero. So we are really finding values of x where the polynomial equals -∞.
- But there’s a problem: **any polynomial with a constant term doesn’t have any roots**! If you add a constant term, your polynomial effectively is min(constant, therestofthepolynomial), which can never be -∞.

Roots in tropical algebra

- Remember what we can do in normal algebra once we get a root. Say I tell you that 9 is a root of $f(x) = x^3 - 8x^2 - 51x + 378$, then you can factor out $(x-9)$ leaving you with $f(x) = (x-9)(x^2 + x - 42)$.
- We can apply this to tropical algebra as well! Say I have a tropical polynomial p, then **r is a root of p if you can factor out $p(x) = (x \oplus r)q(x)$ where q is another polynomial.**
- Now how do we find this? One easy way to find the roots in normal algebra is to graph the polynomial. Maybe we can try that here?

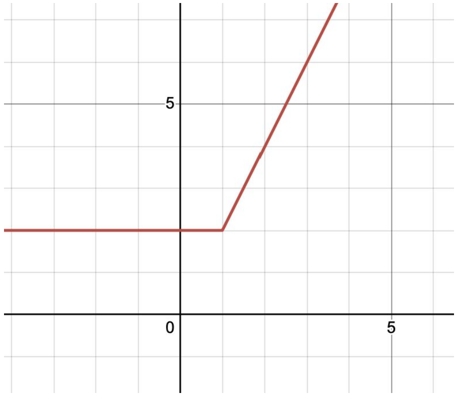
Graphing to find roots

- Let's take the function (x⊕a)(x⊕b). By the definition of root, the roots should be a and b. Let's say a = 1 and b = 4 and look at the graph:



- Note that the roots are just the inflection points of the graph!

- One possibility is if a and b are the same (say they both equal 1). Then the graph looks like:



- We can tell that the root at a has multiplicity 2 because the slope goes from 0 to 2, a jump of +2.

Vectors

- If you have worked with vectors, you know that adding two vectors means just adding all the elements, and multiplying a vector by a scalar means multiplying the scalar by all the elements. In the same way, **we can tropically add two vectors, or multiply a vector by a scalar.**

$$\begin{pmatrix} 4 \\ -1 \\ 2 \\ 8 \end{pmatrix} \oplus \begin{pmatrix} 3 \\ 5 \\ 0 \\ 12 \end{pmatrix} = \begin{pmatrix} \min\{4,3\} \\ \min\{-1,5\} \\ \min\{2,0\} \\ \min\{8,12\} \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 8 \end{pmatrix}$$

$$3 \odot \begin{pmatrix} 4 \\ -1 \\ 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 3+4 \\ 3-1 \\ 3+2 \\ 3+8 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 5 \\ 11 \end{pmatrix}$$

Matrices

- We can also perform matrix multiplication tropically:

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 7 \\ 3 & -6 \end{pmatrix} = \begin{pmatrix} 1 \oplus 1 & -1 \oplus -7 \\ 2 \oplus 3 & 0 \oplus -6 \end{pmatrix} = \begin{pmatrix} 1 & -7 \\ 2 & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \odot \begin{pmatrix} 1 & 7 \\ 3 & -6 \end{pmatrix} = \begin{pmatrix} 1 \odot 1 \oplus -1 \odot 3 & 1 \odot 7 \oplus -1 \odot -6 \\ 2 \odot 1 \oplus 0 \odot 3 & 2 \odot 7 \oplus 0 \odot -6 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \oplus 2 & 8 \oplus -7 \\ 3 \oplus 3 & 9 \oplus -6 \end{pmatrix} = \begin{pmatrix} 2 & -7 \\ 3 & -6 \end{pmatrix}$$

Determinants

- We can also find determinants of tropical matrices the same way we find normal determinants. Behold:

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 6 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 3 & 4 \\ 8 & 7 \end{pmatrix}.$$

$$\operatorname{tropdet}(A) = (3 \odot 6) \oplus (2 \odot 4) = 9 \oplus 6 = 6$$

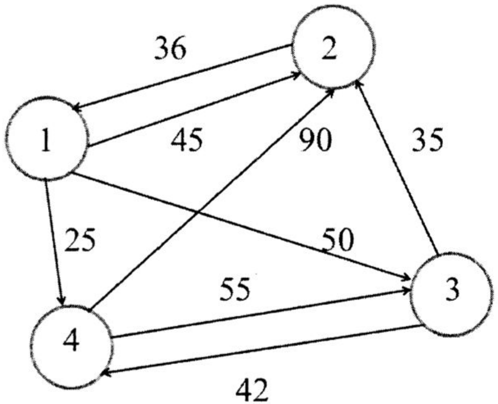
$$\operatorname{tropdet}(B) = (3 \odot 7) \oplus (8 \odot 4) = 10 \oplus 12 = 10$$

So where does this actually get used?

- What's really nice about tropical algebra is that because tropical algebra forms a semiring, **all the math theory we know about addition and multiplication "just works"**, including working with matrices!
- For example, we looked at the determinant of a matrix, which is a very common operation.
- Let's say a company needs to assign n jobs to n workers such that each worker can only do one job, but some workers are better at some jobs than others. Let a_{ij} be the time it takes worker i to do task j. Then **the tropical determinant of the matrix A is the fastest way for all the workers to do all the tasks!**
- To reiterate: common tasks like minimizing cost, time, etc. are just evaluating the tropical determinant.

Example: Shortest/cheapest paths

- Say you are an airline and trying to give someone the cheapest flight path between two cities.
- We'll use min-plus for this situation. A flight between two cities has its cost represented by the number, if there is no direct flight it's listed as inf.



Ticket Cost

$$C = \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & \infty & \infty \\ \infty & 35 & 0 & 42 \\ \infty & 90 & 55 & 0 \end{pmatrix}.$$

Example: cheapest paths (cont.)

- Remember that we can perform matrix multiplication tropically:
- Let's look at this resulting matrix. Notice that the entry in the 3rd row and 1st column corresponds to the path of 3 -> 2 -> 1, and the entry in the 1st row and 3rd column corresponds to the path of 1 -> 3.
- Each entry in the matrix is the cheapest possible path from one city to another in at most 2 flights, since we took the original matrix to the 2nd power.

$$C^2 = \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & \infty & \infty \\ \infty & 35 & 0 & 42 \\ \infty & 90 & 55 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & \infty & \infty \\ \infty & 35 & 0 & 42 \\ \infty & 90 & 55 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & 86 & 61 \\ 71 & 35 & 0 & 42 \\ 126 & 90 & 55 & 0 \end{pmatrix}$$

Example: cheapest paths (cont.)

- We can repeat this process to take C to the 3rd power. In that case, each element of the matrix represents the shortest path between 2 cities using at most 3 flights.
- If we keep taking C to higher powers, it actually stays the same. This makes sense – you only need at most 3 flights to get from one city to another, and adding more flights only increases the total cost.

$$C^3 = C^2 \odot C = \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & 86 & 61 \\ 71 & 35 & 0 & 42 \\ 126 & 90 & 55 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & \infty & \infty \\ \infty & 35 & 0 & 42 \\ \infty & 90 & 55 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 45 & 50 & 25 \\ 36 & 0 & 86 & 61 \\ 71 & 35 & 0 & 42 \\ 126 & 90 & 55 & 0 \end{pmatrix}$$

Example: cheapest paths (cont.)

- So we’ve figured out how to compute the cheapest flight plan between two cities using 1, 2, and 3 flights. The last consideration is that we always want the cheapest path, not the lowest number of flights. If a connecting flight is cheaper than a direct one we’ll prefer that, but if a direct flight is cheaper that one is preferred too.
- Tropical algebra takes care of this for us! Remember in min-plus, $x \oplus y$ is just the minimum of x and y. So we can just add $C \oplus C^2 \oplus C^3$ to find the shortest path between two cities regardless of flight count.
- In fact, this method is the basis of **Dijkstra's Algorithm** and the **Floyd-Warshall Algorithm**, both being famous computer science algorithms that are even covered at NJIT!

Recap

- In the classical sense there are several equivalent definitions for a root of a polynomial, but in tropical algebra some of these don't make sense.
- Tropical algebra translates pretty easily for matrix arithmetic, and the **tropical determinant is a great solution to a minimization problem**!
    - There is also a very efficient method for finding tropical determinants that I skipped
- The connection between tropical matrices and graphs is also very interesting, and easily lends itself to finding shortest paths.
- It's unlikely that you will directly use tropical algebra, but the mathematical outlook in general can be very useful, especially for more algorithms-based problems.